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# Geometric phases in open multi-level systems

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**Abstract.** We analyze the geometric phases in quantum systems coupled to a dissipative environment, when the Hamiltonian of the system, and possibly its coupling to the environment, are slowly varied in time. We find that the coupling to the environment modifies the values of the geometric phases and also induces a geometric contribution to dephasing and relaxation. For a multi-level system with equal level splittings, coupling to the environment makes the dynamics more complex, and we analyze the interplay between various geometric phases in such situations.

## 1.1 Introduction

According to the adiabatic theorem, a quantum system remains in its instantaneous non-degenerate eigenstate, when the Hamiltonian varies slowly enough. If the Hamiltonian is varied along a closed path and returns to its initial value, the initial and final wave functions can differ only by a phase factor. The phase, acquired by a non-degenerate state of the Hamiltonian in addition to the usual dynamical phase $-\int E(t)dt$, is called the Berry phase [1, 2]. It is of geometric nature, i.e., it depends only on geometry of the path but not on the rate and the details of its traversal.

The widespread criterion of adiabaticity requires that the rate of changes of the Hamiltonian be small compared to the energy gap to the neighboring levels. At the same time, any system is coupled, however weakly, to the rest of the universe (which we will refer to as an environment, a bath, or a reservoir), typically with continuous spectrum. This implies that 'true' adiabatic manipulations are impossible, and the adiabatic behavior cannot be approached. On the other hand, a vanishingly weak coupling to the environment cannot change the behavior dramatically. In this context an interesting, important, and significant question is about the Berry phase in an open quantum system.

The Berry phases for open systems were analyzed in earlier theory work [3–8]. In some of this work the behavior of the Berry phases in systems subject to (classical or quantum) noise was analyzed in a language, equivalent to the master equation used below. In part of this work, the visibility of the geometric phases, masked by dephasing, was studied carefully but no modification of the Berry phase was found; e.g., Refs. [3–6] neglected the effect of the finite rate $\partial_t \hat{\mathcal{H}}_0$ on the dissipative rates and the Lamb shift and as a result they found that the BP remains intact. A modification of the Berry phase was found in Ref. [9] for a spin-half in a magnetic field varied along a specific path. Ref. [10] analyzed the variance of the phase and the dephasing due to a random Berry phase, but found no change in its mean value.

Here we use an 'operational' definition of the Berry phase for open systems via measurable quantities. Specifically, we analyze the evolution of the density matrix: for an isolated system the off-diagonal entries of the density matrix acquire phase factors, which contain the dynamic phase and the geometric phase. It can be shown that for an open two-level system the situation is similar: under conditions specified below, the evolution of the off-diagonal entry in the eigenbasis is decoupled from the rest of the density matrix, and as a result of the evolution this entry acquires a factor with a phase, which contains the dynamic and geometric contributions. This Berry phase may differ from its value in an isolated system, and the modification was found in Ref. [9] for a spin-half in an external magnetic field, manipulated in a specific way. Later [11] this modification was calculated for an arbitrary loop, traversed by the tip of the external field, and it was shown that the modification is also of geometric nature, similar to the Berry phase itself. Moreover, it was observed that the modification is complex, or in other words, not only the phase but also the dephasing acquires a geometric contribution. Recently, we have shown that one can also generate geometric phases via adiabatic manipulations of the properties of the noise (that is of the bath of the system-bath coupling) and found this contribution for a two-level system [12].

In this paper we consider the adiabatic dynamics of a multilevel quantum system coupled to an environment. The situation in multi-level systems may be more complex. Indeed, the density matrix has more entries and even in a static field the dynamics of different entries may influence each other. Indeed, the secular approximation, useful for the description of the dynamics and derivation of the Bloch-Redfield equations of motion [13–15], is not applicable to systems with Liouville degeneracies, i.e., when energy gaps between (at least) two pairs of levels are very close [16, 17]. We first analyze the general case without Liouville degeneracies and find the values of Berry phases, accumulated independently between various level pairs. We further analyze the situation with degenerate level splittings and derive the coupled equations of motion for the corresponding coherences, i.e., off-diagonal elements of the density matrix. Furthermore, we study the dynamics of the level occupations, whose evolution is governed by the rate equations. For all these cases we find the dynamical and geometric contributions to the dynamics, that is

to the phases and amplitudes acquired during the evolution. In particular, we find geometric contributions to relaxation. In systems with degenerate levels cyclic adiabatic dynamics may result in more complicated evolutions, coherent unitary transformations of the degenerate subspace for an isolated quantum system. Coupling to an environment may modify this behavior. In the present paper, however, we analyze quantum systems without level degeneracies but with possible Liouville degeneracies, which can make the dynamics non-trivial.

The analysis of geometric phases is of special interest in view of the recent progress in the theoretical and experimental analysis of solid-state, especially superconducting quantum-bit nano-circuits. These systems combine the coherence of the superconducting state with control possibilities of single-electronic and squid devices. They are macroscopic quantum systems and their behavior can be observed with solid-state quantum detectors. On one hand, the level of coherence in the recent experiments is sufficiently high and allows to study even slow adiabatic processes; indeed, recently geometric phases have been demonstrated directly for the first time in solid-state systems [18, 19]. On the other hand, decoherence in these systems is strong enough and its effect on the geometric phases was observed in two-state quantum systems [18] (for earlier direct observations of the Berry phase in various systems see, e.g., Refs. [2, 20–23]). Of special interest are geometric phases and their interplay with decoherence in multi-qubit systems and systems, which include qubits and quantum resonators, i.e., multilevel systems, which are analyzed in the present paper.

The paper is organized as follows: First, we review the known results for the influence of noise on Berry phases in a two-level system. Then we analyze the general structure of geometric phases in multi-level systems. In the further sections we study the geometric phases and dephasing in multi-level system without and with equal level splittings in the energy spectrum. In Section 1.5 we analyze geometric contributions to relaxation.

## 1.2 Berry phase for a two-level system

Before considering the structure and values of geometric phases for a multi-level system we review the results of Refs. [11, 12, 24, 25] for the noise contribution to the Berry phase in a two-level system.

The full Hamiltonian of a two-level spin-half system and its noisy environment reads

$$\hat{\mathcal{H}} = -\frac{1}{2}\mathbf{B}\hat{\boldsymbol{\sigma}} - \frac{1}{2}\hat{X}\mathbf{n}\hat{\boldsymbol{\sigma}} + \hat{\mathcal{H}}_{\rm bath}, \tag{1.1}$$

where the three terms pertain correspondingly to the spin, spin-environment coupling, and the environment. The fast stationary fluctuating quantity $\hat{X}(t)$ represents noise, and $\mathbf{n}$ is an adiabatically varying dimensionless vector, indicating the direction and the power of fluctuations. The Berry phase can be

measured as (a contribution to) the phase of rotation of the spin component orthogonal to **B**. In other words, the off-diagonal (in the eigenbasis) entry of the density matrix as a result of the evolution is multiplied by a factor:

$$\rho_{\uparrow\downarrow}(t) = \rho_{\uparrow\downarrow}(0) e^{i\int_0^t (B+i\Gamma)dt + i(\Phi^0+\delta\Phi)}, \tag{1.2}$$

where the "phase" in the exponent is complex, that is it also describes the change in amplitude. Here the first term in the exponent, $\int (B+i\Gamma)dt$, gives the dynamical phase, where

$$\Gamma = i\left(2|n_+|^2 \int \frac{d\Omega}{2\pi}\frac{S(\Omega)}{\Omega - B + i0} + n_z^2 \int \frac{d\Omega}{2\pi}\frac{S(\Omega)}{\Omega + i0}\right) \tag{1.3}$$

gives the dephasing rate ($\mathrm{Re}\,\Gamma$) and modification of the level splitting ($\mathrm{Im}\,\Gamma$) by the environment. The second term in Eq. (1.2) gives the Berry phase, $\Phi_0$ being the conventional Berry phase for an isolated system and the contribution of the noise to the Berry phase is

$$\begin{aligned}\delta\Phi = &\int \left(i\frac{S(0)}{B} - \frac{1}{2}\int \frac{d\Omega}{2\pi}\frac{S(\Omega)(3B-2\Omega)}{B(\Omega - B + i0)^2}\right)\frac{\mathbf{nB}}{B}\frac{\mathbf{n}(\mathbf{B}\times d\mathbf{B})}{B^2}\\ &-\frac{1}{2}\int\left(\int \frac{d\Omega}{2\pi}\frac{S(\Omega)}{(\Omega - B + i0)^2}\right)\frac{\mathbf{B}(\mathbf{n}\times d\mathbf{n})}{B}\end{aligned} \tag{1.4}$$

with integration along the path of the varying fields **B** and **n**. Here $S(\Omega)$ is the (symmetrized) noise power spectrum of the fluctuating field $\hat{X}(t)$. In writing Eq. (1.4) we omitted the non-universal "boundary phase", determined by details of the initial preparation and the final read-out, and the integral of a full derivative, which vanishes for a closed loop [12].

The first term on the rhs of Eq. (1.4) describes the modification of the Berry phase by the noise [11] and arises, when the varying controlled field **B** rotates about the direction of the fluctuating field **n**. The second term gives the contribution due to rotation of the noise about the controlled field [12].

Below we find the generalization of Eq. (1.4) to the case of a multi-level system.

## 1.3 Structure of Berry phases in a multi-level system

Geometric phases in a multi-level system may be calculated by similar means. The Hamiltonian of a multilevel quantum system, weakly coupled to a reservoir, reads

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0(t) + \hat{X}\hat{V}(t) + \hat{\mathcal{H}}_{\mathrm{bath}}, \tag{1.5}$$

where $\hat{\mathcal{H}}_0(t)$ is the Hamiltonian of the quantum system, which is varied in time via control parameters[4] $\hat{\mathcal{H}}_{\mathrm{bath}}$ governs the dynamics of the bath. The

[4] Typically, in the discussion of geometric phases and adiabatic manipulations, one refers explicitly to a set of control parameters **R** and discusses paths and the

second term describes the coupling between the system and the bath, we take in the form $\hat{X}\hat{V}$, where the field $\hat{X}$ of the bath describes the fluctuations and the operator $\hat{V}$ of the quantum system controls the coupling. We assume that $\hat{\mathcal{H}}_0$ and $\hat{V}$ are varied slowly in time. For later use we represent the Hamiltonian $\hat{\mathcal{H}}_0$ in terms of its (time-dependent) eigenstates and eigenenergies: $\hat{\mathcal{H}}_0 = \sum_k E_k(t) \left|k_t\right\rangle \left\langle k_t\right|$ (below we often omit the subindex $t$).

Let us recall that the concept of the Berry phase [1, 2, 26] is meaningful only for closed paths: During adiabatic evolution (without degeneracies and level crossings) the system, initially in an eigenstate $|k_0\rangle$, remains in an instantaneous eigenstate of the Hamiltonian $\hat{\mathcal{H}}_0(t)$. If finally, at $t = t_P$, the controlled Hamiltonian assumes its initial value[5] (i.e., the Hamiltonian $\hat{\mathcal{H}}_0(t)$ was varied over a closed path), the final state may differ from the initial state $|k_0\rangle$ only by a phase factor, and one can separate this phase into the dynamic and Berry's contributions. At intermediate times, when $\hat{\mathcal{H}}_0(t) \neq \hat{\mathcal{H}}_0(0)$, the notion of the relative phase is ambiguous.

Instead of the analysis of the relative phase of the components of the wave function one can equivalently follow the evolution of (the off-diagonal elements of) the density matrix. For quantum systems coupled to their environments the wave function is ill-defined, and one in fact has to analyze the density matrix. As we find below, under certain generic conditions (and, in particular, for weak noise) the evolution of an off-diagonal entry may be decoupled from the rest of the density matrix; however, during the evolution such an entry gets multiplied by a factor, which is not purely a phase factor, but also changes the amplitude (dephasing or decay of coherence). Let us remark that this suppression of amplitude (which also contains dynamic and geometric part in the adiabatic limit) is well-defined also at intermediate times, since it is not influenced by the phase uncertainty.

In connection with this discussion let us remark that there is certain freedom in the choice of the phase factors of the eigenstates $|k\rangle$, which however does not influence the final results. The gauge invariance w.r.t. this choice of the phases imposes constraints on the expressions for the geometric phases.

### 1.3.1 Berry phases in an isolated system

Let us first consider a closed coherent quantum system, that is a system, which does not interact with its environment. The analysis of the evolution is simplified by a transformation to the instantaneous eigenbasis: consider the transformation $U$, which maps a fixed basis (e.g., the eigenbasis of $\hat{\mathcal{H}}_0$

geometry of loops in the parameter space. Here, for brevity of notation, we just discuss explicit time dependencies $\hat{\mathcal{H}}_0(t)$ and consider paths in the space of the Hamiltonians (the **B**-space for two-level system); one can say that the space of Hamiltonians serves as the (natural) parameter space.

[5] It is sufficient that the eigenbasis of $\hat{\mathcal{H}}_0(t_P)$ is the same as at $t = 0$. By analogy with the spin-half case, one can say that the "direction" of the Hamiltonian should return to its initial value (cf. [12]).

at $t = 0$) to the eigenbasis at time $t$, $U : |k_0\rangle \to |k_t\rangle$. Then we find that the evolution of the transformed wave function $\psi' = U^\dagger \psi$ is described by the Schrödinger equation with the Hamiltonian $\hat{\mathcal{H}}'_0 = U^\dagger \hat{\mathcal{H}}_0 U - iU^\dagger \dot{U}$. The first term is diagonal in the basis $|k_0\rangle$, and for non-degenerate levels in the adiabatic limit the second term is a weak perturbation. The matrix elements of this perturbations are $\langle k_0| - iU^\dagger \dot{U} |l_0\rangle = -i \langle k| \partial_t |l\rangle$.

The effect of the perturbation for a quantum system without degenerate levels is two-fold: it modifies the energy levels:

$$\delta E_k = E'_k - E_k = -i \langle k| \partial_t |k\rangle \ , \tag{1.6}$$

and the eigenstates:

$$|k'\rangle = |k_0\rangle - i \sum_{l \neq k} \frac{\langle l| \partial_t |k\rangle}{E_k - E_l} |l_0\rangle \ . \tag{1.7}$$

This determines the evolution operator (for the transformed wave function $\psi'$ and hence for $\psi$), which is diagonal in this basis with the eigenvalues $\exp[-i \int dt E'_k(t)]$. Note that the corrections to the eigenstates (1.7) modify the evolution operator only slightly, whereas the corrections to the eigenenergies (1.6) are multiplied by $t$ in the exponents and result in much stronger changes of the evolution at growing $t$. (In particular, the possible difference between the primed and non-primed bases at the initial and final moments is a negligible effect.)

For a system with level degeneracies, the off-diagonal matrix elements of the perturbation $-iU^\dagger \dot{U}$ between degenerate states are also relevant. In fact, the evolution in each degenerate subspace is determined by the projection of this perturbation onto this subspace, and this may result in arbitrary unitary holonomic transformations in this subspace [26]. In the current paper we consider only systems without degenerate levels.

### 1.3.2 Influence of fluctuations

Our analysis of geometric phases and dephasing in an open system is based on the Bloch-Redfield approach; we derive a markovian master equation of motion for the reduced density matrix of the quantum system, which is coupled to a reservoir (see, e.g., Refs. [14–16,27] for the derivation and the discussion). To take into account the slow variations of the Hamiltonian of the system, $\hat{\mathcal{H}}_0$, and the effect of the environment, $\hat{V}$, we perform the derivation in the primed representation (i.e., in the primed basis). We assume the following conditions for the time scales involved: $\tau_c, \Delta E^{-1} \ll t_P \ll T_2$, where $\tau_c$ is the noise correlation time, $T_2$ is the dephasing time scale (the decay time of 'coherences', i.e. off-diagonal elements of the density matrix), $\Delta E$ is energy gap in the spectrum of $\hat{\mathcal{H}}_0$. This implies, in particular, that the noise is weak and short-correlated [16], and that on the time scale $t_P$ of the evolution the

noise correlations are local and the coherence is not destroyed completely (and thus the phase information can be detected). In fact, such conditions should be specified independently for different matrix elements since the gaps to the neighboring levels, decay times, and correlation times of various noise components may differ. Moreover, for some matrix elements the decoherence may dominate over the coherent evolution, changing the character of the dynamics (cf. [28]). Here we want to understand the influence of the noise and do not aim at a full analysis of all possible regimes of behavior, and thus would limit ourselves to the case when the conditions specified hold uniformly for all relevant matrix elements. We further introduce the notation for the typical scale of variations, $\omega \sim 1/t_P$, and the typical adiabaticity parameter is $\omega/\Delta E$.

We analyze the phases accumulated by the system (and the dephasing) between times 0 and $t_P$. Detection of this phase may involve preparation of the initial state (e.g., a superposition of various eigenstates to facilitate observation of the relative phases) and the final direct or indirect measurement. We do not specify details of these events, and hence neglect their contributions to the evolution (boundary effects, cf. Refs. [9, 12].

In general, the Bloch-Redfield equations couple all elements of the density matrix: $\partial\rho_{mn} = -i(E_m - E_n)\rho_{mn} - \sum_{kl} \Gamma^{kl}_{mn}\rho_{kl}$. If we consider the second term as a perturbation, we find that to the leading order the time dependence is $\rho_{mn} \propto \exp(-i(E_m - E_n)t)$. Then the rotating-wave approximation shows that only the coupling between matrix elements $\rho_{mn}$ with equal level splittings $E_{mn} \equiv E_m - E_n$ is relevant, and the influence of $\rho_{kl}$ onto $\rho_{mn}$ with a different level splitting averages out on the times scale $|E_{kl} - E_{mn}|^{-1}$. In particular, when all the level splittings are different, each off-diagonal entry $\rho_{mn}$ evolves on its own (Section 1.4.1). If two or more level splittings coincide, the corresponding matrix elements follow joint evolution (Section 1.4.2). In particular, the diagonal elements $\rho_{nn}$ 'correspond' to the zero level splitting ($E_n - E_n = 0$), and hence their relaxational dynamics are coupled (Section 1.5).

As indicated above, it is convenient to perform the analysis in the primed basis. In this basis the matrix elements of the fluctuating field are:

$$\delta V_{mn} = V_{n'm'} - V_{nm} = -i \sum_{k\neq n} \frac{\langle n| \partial_t |k\rangle}{E_n - E_k} V_{km} - i \sum_{k\neq m} \frac{\langle k| \partial_t |m\rangle}{E_m - E_k} V_{nk} \,. \quad (1.8)$$

## 1.4 Calculation of geometric phases and dephasing in multi-level systems

### 1.4.1 Multi-level system with non-degenerate energy splittings

Consider first a quantum system without Liouville degeneracies, that is such a system that all level splittings $E_{mn} = E_m - E_n$ are different throughout

the evolution.[6] As we have discussed above, in this situation the dynamics of each off-diagonal element $\rho_{mn}$ decouples from the dynamics of the other matrix elements. The Bloch-Redfield equation of motion for this element can be obtained from the following integro-differential master equation [14,15,27]:

$$(i\partial_t - E_{mn})\rho_{mn}(t) = -i\int_{-\infty}^{t} \left\langle \left[ [|n_t\rangle \langle m_t|, \hat{\mathcal{V}}(t)], \hat{\mathcal{V}}(t_1) \right] \right\rangle dt_1. \qquad (1.9)$$

Here $\mathcal{V}$ describes the effect of the environment on the quantum system (cf. Eq.(1.5)), and we assume that its average over noise realizations vanishes, when the bath is decoupled from the quantum system. The angle brackets in Eq. 1.9 stand for averaging over noise realizations and the state of the system: $\langle \dots \rangle = \mathrm{tr}(\dots \rho)$. In our calculations, we take the perturbation in the form

$$\hat{\mathcal{V}} = \hat{X}\hat{V}\,, \qquad (1.10)$$

where $\hat{X}$ is the fluctuating field of the bath, and $\hat{V}$ is an operator of the quantum system. For further analysis, we introduce the matrix elements of the perturbations in the instantaneous eigenbasis:

$$\hat{V} = \sum_{k,l} V_{kl} |k\rangle \langle l| \,. \qquad (1.11)$$

Note that although a shift of $\hat{V}$ by a scalar operator $\propto \hat{1}$ does not influence the level splittings of the quantum system directly, it modifies the average value of $\hat{X}$ due to the response of the bath, and thus also contributes to the dynamics of the quantum system. $\hat{V}$ is hermitian and hence $V_{kl} = (V_{lk})^*$.

After using the Redfield and rotating-wave approximations, we find the markovian equation of evolution for $\rho_{mn}$ in the form:

$$\partial_t \rho_{mn} = \left( -iE_{mn} - \Gamma_{mn} + i(\dot{\Phi}^{0,mn}_{\mathrm{BP}} + \delta\dot{\Phi}^{mn}_{\mathrm{BP}}) \right) \rho_{mn} \,. \qquad (1.12)$$

where the first term describes the dynamic phase for a coherent system, $\Gamma_{mn}$ describes the dynamical effects of the bath (the dephasing and the Lamb shift), and for brevity we refer to it as the dephasing rate; $\dot{\Phi}^{0,mn}_{\mathrm{BP}} = -(\delta E_m - \delta E_n) = i(\langle m| \partial_t |m\rangle - \langle n| \partial_t |n\rangle)$ (cf. Eq. (1.6)) describes the contribution to the Berry phase in a closed system, and $\delta\dot{\Phi}_{\mathrm{BP}}$ describes the modification of the Berry phase by the environment. The latter contribution is complex and contains the modification of the geometric phase and also the geometric dephasing.

The dynamical dephasing rate is

$$\Gamma_{mn} = -i\int \frac{d\Omega}{2\pi} \left( \sum_k \left[ \frac{S_c(\Omega)|V_{mk}|^2}{\Omega - E_{mk} - i0} + \frac{S_c(-\Omega)|V_{nk}|^2}{\Omega + E_{nk} - i0} \right] - V_{mm}V_{nn}\frac{2S(\Omega)}{\Omega - i0} \right). \qquad (1.13)$$

[6] In fact, we consider a pair of levels $m$, $n$, such that $E_{mn}$ is different from all other level splittings.

Here $S_c(\Omega)$ is the Fourier image of the noise correlator $S_c(t-t_1) = \langle \hat{X}(t)\hat{X}(t_1)\rangle$, and the noise power $S(\Omega)$ is the symmetrized correlator, i.e., $S(\Omega) = (S_c(\Omega) + S_c(-\Omega))/2$. Notice that $S_c(\Omega)$ is real, $(S_c(\Omega))^* = S_c(\Omega)$.

The noise-induced geometric phase emerges from the following expression:

$$
\begin{aligned}
i\delta\dot{\Phi}^{mn}_{\rm BP} = i\int \frac{d\Omega}{2\pi}\Bigg\{ & \\
& \sum_k \left[\frac{S_c(\Omega)\,\delta(|V_{mk}|^2)}{\Omega - E_{mk} - i0} + \frac{S_c(-\Omega)\,\delta(|V_{nk}|^2)}{\Omega + E_{nk} - i0}\right] - \frac{2S(\Omega)}{\Omega - i0}\delta(V_{mm}V_{nn}) \\
+ & \sum_k \left[\frac{|V_{mk}|^2 S_c(\Omega)}{(\Omega - E_{mk} - i0)^2}\delta E_{mk} - \frac{|V_{nk}|^2 S_c(-\Omega)}{(\Omega + E_{nk} - i0)^2}\delta E_{nk}\right] \\
+ & \sum_k \left[\frac{i|V_{mk}|^2 S_c(\Omega)}{(\Omega - E_{mk} - i0)^3}\dot{E}_{mk} - \frac{i|V_{nk}|^2 S_c(-\Omega)}{(\Omega + E_{nk} - i0)^3}\dot{E}_{nk}\right] \\
+ & \sum_k \left[\frac{iS_c(\Omega)}{(\Omega - E_{mk} - i0)^2}V_{mk}\dot{V}_{km} + \frac{iS_c(-\Omega)}{(\Omega + E_{nk} - i0)^2}\dot{V}_{nk}V_{kn}\right] \\
- & \frac{iS_c(\Omega)}{(\Omega - i0)^2}\dot{V}_{mm}V_{nn} - \frac{iS_c(-\Omega)}{(\Omega - i0)^2}V_{mm}\dot{V}_{nn}\Bigg\}, \qquad (1.14)
\end{aligned}
$$

where $\dot{V}_{kl}$ is a notation for $\partial_t(V_{kl}) = \partial_t(\langle k|\,\hat{V}\,|l\rangle)$, and this time derivative includes the effects of the time dependence of the perturbation $\hat{V}$ ('rotation of noise') as well as of the basis states $|k\rangle$, $|l\rangle$ (variation of the Hamiltonian $\hat{\mathcal{H}}_0(t)$). Further, $\dot{E}_{kl}$ is the time derivative of the level splitting, i.e., of the difference between the $k$th and $l$th eigenenergies of $\hat{\mathcal{H}}_0(t)$. Finally, $\delta V_{kl}$ and $\delta E_{kl}$ arise due to the difference between the primed and non-primed frames: $\delta V_{kl} = V'_{kl} - V_{kl}$ as given by Eq. (1.8) and $\delta E_{kl} = \delta E_k - \delta E_l$, where $\delta E_k = -i\,\langle k|\,\partial_r\,|k\rangle$ as given by Eq. (1.6).

Let us discuss these results (1.13) and (1.14). Note that the (dynamical) dephasing rate can be expressed via the outgoing transition rates from levels $m$, $n$ to other levels and the rate of pure dephasing (cf. Refs. [16, 17, 29]):

$$
\mathrm{Re}\,\Gamma_{mn} = \frac{1}{2}\sum_{k\neq n}\Gamma^{\rm rel}_{k\leftarrow n} + \frac{1}{2}\sum_{k\neq m}\Gamma^{\rm rel}_{k\leftarrow m} + \Gamma^{\varphi}_{mn}, \qquad (1.15)
$$

where $\Gamma^{\rm rel}_{j\leftarrow i} = |V_{ij}|^2 S_c(E_{ij})$ is the relaxation rate $|i\rangle \to |j\rangle$, and $\Gamma^{\varphi}_{mn} = \frac{1}{2}(V_{mm} - V_{nn})^2 S(0)$ is the pure-dephasing rate of $\rho_{mn}$.

As for the modification of the level splitting, the "Lamb shift", it can be presented as:

$$
\mathrm{Im}\,\Gamma_{mn} = \delta E_m - \delta E_n\,, \qquad (1.16)
$$

where $\delta E_m = \sum_k \delta E^k_m$, and

$$\delta E_i^j = |V_{ij}|^2 P.V. \int \frac{d\Omega}{2\pi} \frac{S_c(\Omega)}{E_{ij} - \Omega} \tag{1.17}$$

is the modification of the energy of level $i$ by the coupling (virtual transitions) to level $j$. Note that for a two-level system the quantities $\delta E_1^2 - \delta E_2^1$ and $\mathrm{Re}\, \Gamma_{12}$ depend only on the symmetrized noise correlator (cf. Ref. [11, 24]).

Now let us discuss the geometric contribution (1.14). The first two lines, which can be denoted as $-\delta\Gamma_{mn}$, show the difference of the expression (1.13) in the primed and non-rimed frames. In the fourth line one can replace $V_{mk}\dot{V}_{km}$ by $(\frac{1}{2}\partial_t + i\dot{\Phi}_{km})|V_{mk}|^2$, where $\Phi_{km}$ is the phase of $V_{km} = |V_{km}|e^{i\Phi_{km}}$, and similarly $\dot{V}_{nk}V_{kn} = (\frac{1}{2}\partial_t + i\dot{\Phi}_{nk})|V_{nk}|^2$. Then the first terms here, $\propto \partial_t|V|^2$, form a full derivative together with the third line (which contains $\dot{E}_{mk}$), and this contribution vanishes after integration over a closed loop; whereas the terms with $\dot{\Phi}$'s can be combined with the second line, and these time derivatives of phases and $\delta E$'s enter only in combinations $\delta E_{mk} + i\dot{\Phi}_{mk}$. This fact is a consequence of the gauge invariance w.r.t. to multiplication of the basis states $|k_t\rangle$ by arbitrary time-dependent phase factors.

### 1.4.2 System with equal energy splittings

Consider now the simplest example of a system with a Liouville degeneracy, i.e., a system with two pairs of energy levels, $|n_1\rangle$, $|m_1\rangle$ and $|n_2\rangle$, $|m_2\rangle$, with equal splittings: $E_{n_1m_1} = E_{n_2m_2} \equiv E_{nm}$ (Fig. 1.1). This example allows one to understand also the evolution in more complicated situations, for instance, with a larger set of level pairs with the same energy splitting. We assume for simplicity that the splittings coincide at all times and that all other energy splittings differ from these two.

Using the Bloch-Redfield approach, we find the joint equation of motion for the off-diagonal matrix elements of the density matrix to the leading order in the adiabatic parameter:

$$\frac{d}{dt}\begin{pmatrix}\rho_{m_1n_1}\\ \rho_{m_2n_2}\end{pmatrix} = \Bigg[ -iE_{m_1n_1} + i\begin{pmatrix}C^1 & 0\\ 0 & C^2\end{pmatrix} + \begin{pmatrix}-\Gamma^{11} & \Gamma^{12}\\ \Gamma^{21} & -\Gamma^{22}\end{pmatrix} + i\begin{pmatrix}a^{11} & a^{12}\\ a^{21} & a^{22}\end{pmatrix}\Bigg]\begin{pmatrix}\rho_{m_1n_1}\\ \rho_{m_2n_2}\end{pmatrix}. \tag{1.18}$$

Here the quantities $C^{1,2} = \dot{\Phi}_{\mathrm{BP}}^{0,m_in_i}$ give rise to the standard Berry phases for the two off-diagonal matrix elements; the rates $\Gamma$ describe the contribution of the noise to the dynamical phase (and dephasing), whereas $a$'s are responsible for the geometric contributions. The diagonal entries $\Gamma^{ii}$ of these matrices are given by the expressions $\Gamma_{m_in_i}$ from the previous section, Eq. (1.13), and $a^{ii}$ are given by $\dot{\delta\Phi}_{\mathrm{BP}}^{m_in_i}$ from the previous section, Eq. (1.14). Note that since Eq. (1.18) is a matrix equation, we have to keep full time derivatives, e.g., in the $a$-matrix, even for evolution along a closed loop.

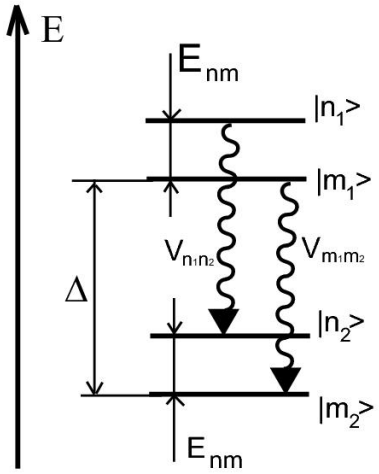


**Fig. 1.1.** A scheme of energy levels with two pairs having equal splittings. The relaxation processes from one pair to the other make the corresponding elements of the density matrix evolving dependently.

For the off-diagonal entries we find:

$$\Gamma^{12} = -iV_{m_1m_2}V_{n_2n_1}\int\frac{d\Omega}{2\pi}\left(\frac{S_c(\Omega)}{\Omega+\Delta-i0}+\frac{S_c(-\Omega)}{\Omega-\Delta-i0}\right) \tag{1.19}$$

$$= V_{m_1m_2}V_{n_2n_1}S_c(-\Delta)\,. \tag{1.20}$$

Here we have introduced the notation $\Delta \equiv E_{m_1m_2} = E_{n_1n_2}$.

The geometric contributions are controlled by

$$a^{12} = -i\delta(V_{m_1m_2}V_{n_2n_1})S_c(-\Delta) \tag{1.21}$$

$$-\int\frac{d\Omega}{2\pi}\Bigg\{V_{m_1m_2}V_{n_2n_1}\left[\frac{S_c(\Omega)}{(\Omega+\Delta-i0)^2}\delta E_{n_2n_1}-\frac{S_c(-\Omega)}{(\Omega-\Delta-i0)^2}\delta E_{m_2m_1}\right]$$

$$-i\frac{S_c(\Omega)\dot{V}_{m_1m_2}V_{n_2n_1}}{(\Omega+\Delta-i0)^2}-i\frac{S_c(-\Omega)V_{m_1m_2}\dot{V}_{n_2n_1}}{(\Omega-\Delta-i0)^2}$$

$$-i\frac{S_c(\Omega)V_{m_1m_2}V_{n_2n_1}}{(\Omega+\Delta-i0)^3}\dot{\Delta}-i\frac{S_c(-\Omega)V_{m_1m_2}V_{n_2n_1}}{(\Omega-\Delta-i0)^3}\dot{\Delta}\Bigg\}\,. \tag{1.22}$$

$\Gamma^{21}$ and $a^{21}$ are given by similar expressions with the substitution $1 \leftrightarrow 2$. In the case of an arbitrary number $N$ of equal splittings in the energy spectrum, we would arrive at non-diagonal $N \times N$ matrices $\Gamma^{ij}$ and $a^{ij}$, given by similar expressions. Note also that the evolution in Eq. (1.18) is typically dominated by the coherent terms $C^{1,2}$; when these coherent terms dominate over the noise-induced contributions $\check{\Gamma}$ and $\check{a}$, the off-diagonal entries can be neglected

as they provide only small corrections (unless $C^1$ and $C^2$ coincide or are very close during the evolution).

Notice that $\Gamma^{21}$ (the 'transition rate between $\rho_{m_1n_1}$ and $\rho_{m_2n_2}$') is related to the transition rates between the levels $m_1$ and $m_2$ and between the levels $n_1$ nd $n_2$, both with the same splitting $\Delta$:

$$|\Gamma^{21}| = \sqrt{\Gamma^{\rm rel}_{m_2\leftarrow m_1}\Gamma^{\rm rel}_{n_2\leftarrow n_1}} \tag{1.23}$$

and similarly $|\Gamma^{12}| = \sqrt{\Gamma^{\rm rel}_{m_1\leftarrow m_2}\Gamma^{\rm rel}_{n_1\leftarrow n_2}}$, in the lowest-order of the expansion in $V$. In particular, $|\Gamma^{21}| \leq \frac{1}{2}(\Gamma^{\rm rel}_{m_1\leftarrow m_2} + \Gamma^{\rm rel}_{n_1\leftarrow n_2}) \leq \mathrm{Re}\,\Gamma_{m_1n_1} = \mathrm{Re}\,\Gamma^{11}$.

Since the noise couples the evolution of two off-diagonal entries, by measuring only one of them, one can deduce information about both Berry phases. Consider, for example, two level pairs, each of which, in the absence of fluctuations, acquires a relative geometric phase uniformly in time, with the rates $C^{1,2}$. Let us assume further that the relaxation induces transitions only in one direction, so that $\Gamma^{21} \neq 0$ but $\Gamma^{12} = 0$. Then one finds

$$\begin{aligned}\rho_{m_2n_2}(t) = e^{i\Phi^2_{\rm BP}-\Gamma^{22}t}\left[\rho_{m_2n_2}(0) + \frac{\Gamma^{21}}{i(C^2-C^1)+(\Gamma^{11}-\Gamma^{22})}\rho_{m_1n_1}(0)\right]\\ - e^{i\Phi^1_{\rm BP}-\Gamma^{11}t}\frac{\Gamma^{21}}{i(C^2-C^1)+(\Gamma^{11}-\Gamma^{22})}\rho_{m_1n_1}(0)\,,\end{aligned} \tag{1.24}$$

where $\Phi^i_{\rm BP} = C^i t$ is the Berry phase, which would be acquired by $\rho_{m_in_i}$ in the absence of fluctuations ($i = 1, 2$). In particular, the effect of the Berry phase $\Phi^1_{\rm BP}$ may be observed in $\rho_{m_2n_2}$ at $t \sim 1/\Gamma \sim C^i \sim C^1 - C^2$.

## 1.5 Geometric relaxation

Let us now consider the dynamics of the diagonal entries of the density matrix, that is transitions between the levels (relaxation / excitation depending on the direction in energy). All these entries $\rho_{nn}$ correspond to zero energy difference, $E_n - E_n = 0$, and their dynamics, in general, is coupled. Since we consider systems without degenerate levels, no other entries are coupled with the diagonal elements in the rotating-wave approximation. Thus, the dynamics of the diagonal entries is described by a rate equation. From the Bloch-Redfield formalism, one finds the transition rates between two distinct levels $k$ and $m$ (from $k$ to $m$):

$$\Gamma^{\rm rel}_{m\leftarrow k} = |V_{mk}|^2 S_c(E_{km})\,. \tag{1.25}$$

The first adiabatic correction to this expression is

$$\gamma^{\rm rel}_{m\leftarrow k} = \delta(|V_{mk}|^2)S_c(E_{km}) + |V_{mk}|^2 S_c'(E_{km})\delta E_{km}$$

$$+\int \frac{d\Omega}{2\pi}\left[\left(\frac{S_c(\Omega)|V_{mk}|^2}{(\Omega-E_{km}-i0)^3}-\frac{S_c(-\Omega)|V_{mk}|^2}{(\Omega+E_{km}-i0)^3}\right)\dot{E}_{km}\right.$$
$$\left.+\frac{S_c(\Omega)}{(\Omega-E_{km}-i0)^2}\dot{V}_{mk}V_{km}+\frac{S_c(-\Omega)}{(\Omega+E_{km}-i0)^2}V_{mk}\dot{V}_{km}\right], \quad (1.26)$$

and this correction is responsible for the geometric contributions to the relaxational dynamics.

Note that the relaxation rates are real and that the dynamic and geometric contributions to relaxation are gauge-invariant and well-defined for open paths. Similar to geometric phases and geometric dephasing [11] the geometric contribution to relaxation changes sign, when the same path is traversed in the opposite direction (i.e., when the Hamiltonian and / or the parameters of the fluctuations are varied backwards along the same path).

## 1.6 Conclusions

In this paper we have analyzed the influence of fluctuations on the adiabatic evolution in multi-level quantum systems. During adiabatic evolution the off-diagonal entries of the system's density matrix acquire dynamic and geometric phases. We found that the noise modifies the values of the geometric phases and also induces geometric contributions to dephasing and relaxation. In a multi-level system without degenerate levels and without equal energy splittings for different level pairs, the dynamics of different off-diagonal entries are decoupled, whereas in systems with equal level splittings (but no degenerate levels) the equations of motion for several off-diagonal entries may be coupled, which results in more complicated evolutions. Similarly, the dynamics of diagonal entries (occupations of the eigenstates) are coupled and are described by the rate equations, which also contain geometric contributions. We acknowledge useful discussions with A. Shnirman. This work was partially supported by the projects INTAS 05-1000008-7923, MD-4092.2007.2, and the Dynasty foundation.

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